Group classification of steady two-dimensional boundary-layer stagnation-point flow equations

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Abstract

Lie symmetry group method is applied to study the boundary-layer equations for two-dimensional steady flow of an incompressible, viscous fluid near a stagnation point at a heated stretching sheet placed in a porous medium equation. The symmetry group and its optimal system are given, and group invariant solutions associated to the symmetries are obtained. Finally the structure of the Lie algebra symmetries is determined.

Key words: Fluid mechanics; Lie symmetry; Partial differential equation; Incompressible viscous fluid; Stagnation point.

1 Introduction

In fluid dynamics, a stagnation point is a point in a flow field where the local velocity of the fluid is zero. Stagnation points exist at the surface of objects in the flow field, where the fluid is brought to rest by the object. The Bernoulli equation shows that the static pressure is highest when the velocity is zero and hence static pressure is at its maximum value at stagnation points. This static pressure is called the stagnation pressure. The Bernoulli equation applicable to incompressible flow shows that the stagnation pressure is equal to the dynamic pressure plus static pressure. Total pressure is also equal to dynamic pressure plus static pressure so, in incompressible flows, stagnation pressure is equal to total pressure. (In compressible flows, stagnation pressure is also equal to total pressure providing the fluid entering the stagnation point is brought to rest is entropically.)

In physics and fluid mechanics, a boundary layer is that layer of fluid in the immediate vicinity of a bounding surface. In the Earth's atmosphere, the planetary boundary layer is the air layer near the ground affected by diurnal heat, moisture or momentum transfer to or from the surface. On an aircraft wing the boundary layer is the part of the flow close to the wing. The boundary layer effect occurs at the field region in which all changes occur in the flow pattern. The boundary layer distorts surrounding non-viscous flow. It is a phenomenon of viscous forces. This effect is related to the Reynolds number (In fluid mechanics and heat transfer, the Reynold's number is a dimensionless number that gives a measure of the ratio of inertial forces to viscous and, consequently, it quantifies the relative importance of these two types of forces for given flow conditions). Laminar boundary layers come in various forms and can be loosely classified according to their structure and the circumstances under which they are created. The thin shear layer which develops on an oscillating body is an example of a Stokes boundary layer, whilst the Blasius boundary layer refers to the well-known similarity solution for the steady boundary layer attached to a flat plate held in an oncoming unidirectional flow. When a fluid rotates, viscous forces may be balanced by the Coriolis effect, rather than convective inertia, leading to the formation of an Ekman layer. Thermal boundary layers also exist in heat transfer. Multiple types of boundary layers can coexist near a surface simultaneously. The deduction of the boundary layer equations was perhaps one of the most important advances in fluid dynamics. Using an order of magnitude analysis, the well-known governing NavierStokes equations of viscous fluid flow can be greatly simplified

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within the boundary layer. Notably, the characteristic of the partial differential equations (PDE) becomes parabolic, rather than the elliptical form of the full NavierStokes equations. This greatly simplifies the solution of the equations. By making the boundary layer approximation, the flow is divided into an inviscid portion (which is easy to solve by a number of methods) and the boundary layer, which is governed by an easier to solve PDE.

Flow and heat transfer of an incompressible viscous fluid over a stretching sheet appear in several manufacturing processes of industry such as the extrusion of polymers, the cooling of metallic plates, the aerodynamic extrusion of plastic sheets, etc. In the glass industry, blowing, floating or spinning of fibres are processes, which involve the flow due to a stretching surface. Mahapatra and Gupta studied the steady two-dimensional stagnation-point flow of an incompressible viscous fluid over a flat deformable sheet when the sheet is stretched in its own plane with a velocity proportional to the distance from the stagnation-point. They concluded that, for a fluid of small kinematic viscosity, a boundary layer is formed when the stretching velocity is less than the free stream velocity and an inverted boundary layer is formed when the stretching velocity exceeds the free stream velocity. Temperature distribution in the boundary layer is determined when the surface is held at constant temperature giving the so called surface heat flux. In their analysis, they used the finite-differences scheme along with the Thomas algorithm to solve the resulting system of ordinary differential equations.

This paper is concerned with the solution of steady two-dimensional stagnation point flow of an incompressible viscous fluid over a stretching sheet which is placed in a fluid saturated porous medium. Lie-group theory is applied to the equations of motion for determining symmetry reductions of partial differential equations.

2 Lie Symmetries of the Equations

A PDE with $p-$independent and $q-$dependent variables has a Lie point transformations

$$\widetilde{x}_i = x_i + \varepsilon\xi_i(x,u) + \mathcal{O}(\varepsilon^2), \qquad \widetilde{u}_\alpha = u_\alpha + \varepsilon\varphi_\alpha(x,u) + \mathcal{O}(\varepsilon^2),$$

where $\xi_i = \dfrac{\partial \widetilde{x}_i}{\partial \varepsilon}\Big|_{\varepsilon=0}$ for $i = 1, ..., p$ and $\varphi_\alpha = \dfrac{\partial \widetilde{u}_\alpha}{\partial \varepsilon}\Big|_{\varepsilon=0}$ for $\alpha = 1, ..., q$. The action of the Lie group can be considered by its associated infinitesimal generator

$$\mathbf{v} = \sum_{i=1}^{p} \xi_i(x,u)\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \varphi_\alpha(x,u)\frac{\partial}{\partial u_\alpha} \tag{1}$$

on the total space of PDE (the space containing independent and dependent variables). Furthermore, the characteristic of the vector field (1) is given by

$$Q^\alpha(x, u^{(1)}) = \varphi_\alpha(x,u) - \sum_{i=1}^{p} \xi_i(x,u)\frac{\partial u^\alpha}{\partial x_i},$$

and its $n-$th prolongation is determined by

$$\mathbf{v}^{(n)} = \sum_{i=1}^{p} \xi_i(x,u)\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q}\sum_{\sharp J=j=0}^{n} \varphi_\alpha^J(x, u^{(j)})\frac{\partial}{\partial u_J^\alpha},$$

where $\varphi_\alpha^J = D_J Q^\alpha + \sum_{i=1}^{p} \xi_i u_{J,i}^\alpha$. ($D_J$ is the total derivative operator describes in (4)).

The deduction of the boundary layer equations was perhaps one of the most important advances in fluid dynamics. Using an order of magnitude analysis, the well-known governing NavierStokes equations of viscous fluid flow can be greatly simplified within the boundary layer. Notably, the characteristic of the partial differential equations becomes parabolic, rather than the elliptical form of the full NavierStokes equations. This greatly simplifies the solution of the equations. By making the boundary layer approximation, the flow is divided into an inviscid portion (which is easy to solve by a number of methods) and the boundary layer, which is governed by an easier to solve PDE. The

continuity and NavierStokes equations for a two-dimensional steady incompressible flow in Cartesian coordinates are given by

$$\begin{cases} \dfrac{\partial U}{\partial x} + \dfrac{\partial V}{\partial y} = 0, \\ U\dfrac{\partial U}{\partial x} + V\dfrac{\partial V}{\partial y} = P\dfrac{\partial P}{\partial x} + \nu\dfrac{\partial^2 U}{\partial y^2} + \dfrac{\nu}{k}(P-U), \\ U\dfrac{\partial T}{\partial x} + V\dfrac{\partial T}{\partial y} = \alpha\dfrac{\partial^2 T}{\partial y^2}. \end{cases} \tag{2}$$

where U, V, P and T are smooth functions of (x, y), and also U and V are the velocity components along x-axis and y-axis respectively, ν is the kinematic viscosity, k is the permeability of the porous medium, T is the fluid temperature, α is the coefficient of thermal diffusivity. The aim is to analysis the Lie point symmetry structure of the system of steady two-dimensional boundary-layer stagnation-point flow equations.

Let us consider a one-parameter Lie group of infinitesimal transformations (x, y, U, V, P, T) given by

$$\begin{aligned} \widetilde{x} &= x + \varepsilon\xi_1(x, y, U, V, P, T) + \mathcal{O}(\varepsilon^2), & \widetilde{y} &= y + \varepsilon\xi_2(x, y, U, V, P, T) + \mathcal{O}(\varepsilon^2), \\ \widetilde{U} &= U + \varepsilon\eta_1(x, y, U, V, P, T) + \mathcal{O}(\varepsilon^2), & \widetilde{V} &= V + \varepsilon\eta_2(x, y, U, V, P, T) + \mathcal{O}(\varepsilon^2), \\ \widetilde{P} &= P + \varepsilon\eta_3(x, y, U, V, P, T) + \mathcal{O}(\varepsilon^2), & \widetilde{T} &= T + \varepsilon\eta_4(x, y, U, V, P, T) + \mathcal{O}(\varepsilon^2), \end{aligned}$$

where ε is the group parameter. Then one requires that this transformations leaves invariant the set of solutions of the system (2). This yields to the linear system of equations for the infinitesimals $\xi_1(x, y, U, V, P, T), \xi_2(x, y, U, V, P, T)$, $\eta_1(x, y, U, V, P, T), \eta_2(x, y, U, V, P, T), \eta_3(x, y, U, V, P, T)$ and $\eta_4(x, y, U, V, P, T)$. The Lie algebra of infinitesimal symmetries is the set of vector fields in the form of

$$\mathbf{v} = \xi_1\frac{\partial}{\partial x} + \xi_2\frac{\partial}{\partial y} + \eta_1\frac{\partial}{\partial U} + \eta_2\frac{\partial}{\partial V} + \eta_3\frac{\partial}{\partial P} + \eta_4\frac{\partial}{\partial T}.$$

This vector field has the second prolongation

$$\begin{aligned} \mathbf{v}^{(2)} = \mathbf{v} &+ \varphi_1^x\frac{\partial}{\partial x} + \varphi_1^y\frac{\partial}{\partial y} + \varphi_1^{xx}\frac{\partial}{\partial U_{xx}} + \varphi_1^{xy}\frac{\partial}{\partial U_{xy}} + \varphi_1^{yy}\frac{\partial}{\partial U_{yy}} + \varphi_2^x\frac{\partial}{\partial V_x} + \varphi_2^y\frac{\partial}{\partial V_y} + \varphi_2^{xx}\frac{\partial}{\partial V_{xx}} \\ &+ \varphi_2^{xy}\frac{\partial}{\partial V_{xy}} + \varphi_2^{yy}\frac{\partial}{\partial V_{yy}} + \varphi_3^x\frac{\partial}{\partial P_x} + \varphi_3^y\frac{\partial}{\partial P_y} + \varphi_3^{xx}\frac{\partial}{\partial P_{xx}} + \varphi_3^{xy}\frac{\partial}{\partial P_{xy}} + \varphi_3^{yy}\frac{\partial}{\partial P_{yy}} \\ &+ \varphi_4^x\frac{\partial}{\partial T_x} + \varphi_4^y\frac{\partial}{\partial T_y} + \varphi_4^{xx}\frac{\partial}{\partial T_{xx}} + \varphi_4^{xy}\frac{\partial}{\partial T_{xy}} + \varphi_4^{yy}\frac{\partial}{\partial T_{yy}}, \end{aligned} \tag{3}$$

with the coefficients

$$\begin{aligned} \varphi_1^x &= D_x(\varphi_1 - \xi_1U_x - \xi_2U_y) + \xi_1U_{xx} + \xi_2U_{xy}, & \varphi_1^y &= D_y(\varphi_1 - \xi_1U_x - \xi_2U_y) + \xi_1U_{xy} + \xi_2U_{yy}, \\ \varphi_1^{xx} &= D_x^2(\varphi_1 - \xi_1U_x - \xi_2U_y) + \xi_1U_{xxx} + \xi_2U_{xxy}, & \varphi_1^{xy} &= D_xD_y(\varphi_1 - \xi_1U_x - \xi_2U_y) + \xi_1U_{xxy} + \xi_2U_{xxy}, \\ \varphi_1^{yy} &= D_y^2(\varphi_1 - \xi_1U_x - \xi_2U_y) + \xi_1U_{xxy} + \xi_2U_{xyy}, & \varphi_2^x &= D_x(\varphi_2 - \xi_1V_x - \xi_2V_y) + \xi_1V_{xx} + \xi_2V_{xy}, \\ \varphi_2^y &= D_y(\varphi_2 - \xi_1V_x - \xi_2V_y) + \xi_1V_{xy} + \xi_2V_{yy}, & \varphi_2^{xx} &= D_x^2(\varphi_2 - \xi_1V_x - \xi_2V_y) + \xi_1V_{xxx} + \xi_2V_{xxy}, \\ \varphi_2^{xy} &= D_xD_y(\varphi_2 - \xi_1V_x - \xi_2V_y) + \xi_1V_{xxy} + \xi_2V_{xxy}, & \varphi_2^{yy} &= D_y^2(\varphi_2 - \xi_1P_x - \xi_2P_y) + \xi_1P_{xxy} + \xi_2P_{xyy}, \\ \varphi_3^x &= D_x(\varphi_3 - \xi_1P_x - \xi_2P_y) + \xi_1P_{xx} + \xi_2P_{xy}, & \varphi_3^y &= D_y(\varphi_3 - \xi_1P_x - \xi_2P_y) + \xi_1P_{xy} + \xi_2P_{yy}, \\ \varphi_3^{xx} &= D_x^2(\varphi_3 - \xi_1P_x - \xi_2P_y) + \xi_1P_{xxx} + \xi_2P_{xxy}, & \varphi_3^{xy} &= D_xD_y(\varphi_3 - \xi_1T_x - \xi_2T_y) + \xi_1T_{xxy} + \xi_2T_{xxy}, \\ \varphi_3^{yy} &= D_y^2(\varphi_3 - \xi_1T_x - \xi_2T_y) + \xi_1T_{xxy} + \xi_2T_{xyy}, & \varphi_4^x &= D_x(\varphi_4 - \xi_1T_x - \xi_2T_y) + \xi_1T_{xx} + \xi_2T_{xy}, \\ \varphi_4^y &= D_y(\varphi_4 - \xi_1T_x - \xi_2T_y) + \xi_1T_{xy} + \xi_2T_{yy}, & \varphi_4^{xx} &= D_x^2(\varphi_4 - \xi_1P_x - \xi_2P_y) + \xi_1P_{xxx} + \xi_2P_{xxy}, \\ \varphi_4^{xy} &= D_xD_y(\varphi_4 - \xi_1T_x - \xi_2T_y) + \xi_1T_{xxy} + \xi_2T_{xxy}, & \varphi_4^{yy} &= D_y^2(\varphi_4 - \xi_1T_x - \xi_2T_y) + \xi_1T_{xxy} + \xi_2T_{xyy}, \end{aligned}$$

Table 1
Commutation relations of $\mathfrak{g}$

$[\,,\,]$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
$\mathbf{v}_1$	0	0	0	0
$\mathbf{v}_2$	0	0	0	0
$\mathbf{v}_3$	0	0	0	$\mathbf{v}_3$
$\mathbf{v}_4$	0	0	$-\mathbf{v}_3$	0

where the operators D_x and D_y denote the total derivative with respect to x and t:

$$D_x = \frac{\partial}{\partial x} + U_x\frac{\partial}{\partial U} + V_x\frac{\partial}{\partial V} + P_x\frac{\partial}{\partial P} + T_x\frac{\partial}{\partial T}\cdots, \quad D_y = \frac{\partial}{\partial y} + U_y\frac{\partial}{\partial U} + V_y\frac{\partial}{\partial V} + P_y\frac{\partial}{\partial P} + T_y\frac{\partial}{\partial T}\cdots. \tag{4}$$

Using the invariance condition, i.e., applying the second prolongation $\mathbf{v}^{(2)}$, (3), to system (2), and by solving the linear system

$$\begin{cases} \mathbf{v}^{(2)}\Big(\dfrac{\partial U}{\partial x} + \dfrac{\partial V}{\partial y}\Big) = 0, \\ \mathbf{v}^{(2)}\Big(U\dfrac{\partial U}{\partial x} + V\dfrac{\partial V}{\partial y} - P\dfrac{\partial P}{\partial x} - \nu\dfrac{\partial^2 U}{\partial y^2} - \dfrac{\nu}{k}(P-U)\Big) = 0, \qquad (\mathrm{mod}\ (2)) \\ \mathbf{v}^{(2)}\Big(U\dfrac{\partial T}{\partial x} + V\dfrac{\partial T}{\partial y} - \alpha\dfrac{\partial^2 T}{\partial y^2}\Big) = 0, \end{cases}$$

the following system of 112 determining equations yields:

$$2\alpha\frac{\partial^2\xi_2}{\partial T\partial y} + 2V\frac{\partial\xi_2}{\partial T} - \alpha\frac{\partial^2\eta_4}{\partial T^2} = 0, \qquad \alpha V\frac{\partial\xi_1}{\partial U} + \alpha\nu\frac{\partial^2\xi_1}{\partial V\partial y} - \nu V\frac{\partial\xi_1}{\partial V} + 2\nu U\frac{\partial\xi_2}{\partial V} - \alpha U\frac{\partial\xi_1}{\partial U} + \nu U\frac{\partial\xi_1}{\partial U} = 0,$$
$$V\frac{\partial\xi_1}{\partial U} + U\frac{\partial\xi_2}{\partial V} = 0, \qquad U\frac{\partial\xi_1}{\partial V} + V\frac{\partial\xi_1}{\partial V} = 0, \qquad \ldots\ldots \quad 2\frac{\partial^2\xi_2}{\partial U\partial y} - \frac{\partial^2\eta_1}{\partial U^2} = 0, \qquad \frac{\partial^2\xi_1}{\partial U^2} - 2\frac{\partial\xi_2}{\partial^2 V\partial U} = 0.$$

The solution of the above system gives the following coefficients of the vector field $\mathbf{v}$:

$$\xi_1 = C_1, \quad \xi_2 = C_2, \quad \eta_1 = 0, \quad \eta_2 = 0, \quad \eta_3 = T, \quad \eta_4 = 0,$$

where C_1 and C_2 are arbitrary constants, thus the Lie algebra $\mathfrak{g}$ of the steady two-dimensional boundary-layer stagnation-point flow equations is spanned by the four vector fields

$$\mathbf{v}_1 = \frac{\partial}{\partial x}, \quad \mathbf{v}_2 = \frac{\partial}{\partial y}, \quad \mathbf{v}_3 = \frac{\partial}{\partial T}, \quad \mathbf{v}_4 = T\frac{\partial}{\partial T},$$

which $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ are translation on x, u and T, $\mathbf{v}_4$ is scaling on T. The commutation relations between these vector fields is given by the table 1, where entry in row i and column j representing $[\mathbf{v}_i, \mathbf{v}_j]$.

The one-parameter groups G_i generated by the base of $\mathfrak{g}$ are given in the following table.

$$G_1 : (x+\varepsilon, y, U, V, P, T), \quad G_2 : (x, y+\varepsilon, U, V, P, T), \quad G_3 : (x, y, U, V, P, T+\varepsilon), \quad G_4 : (x, y, U, V, P, Te^{\varepsilon}).$$

Since each group G_i is a symmetry group and if $U = f(x,y), V = g(x,y), P = h(x,y)$ and $T = r(x,y)$ are solutions

of the system (2), so are the functions

$$\begin{array}{llll}
U_1 = f(x+\varepsilon, y), & U_1 = f(x, y+\varepsilon), & U_1 = f(x,y), & U_1 = f(x,y), \\
V_1 = g(x+\varepsilon, y), & V_1 = g(x, y+\varepsilon), & V_1 = g(x,y), & V_1 = g(x,y), \\
P_1 = h(x+\varepsilon, y), & P_1 = h(x, y+\varepsilon), & P_1 = h(x,y), & P_1 = h(x,y), \\
T_1 = r(x+\varepsilon, y), & T_1 = r(x, y+\varepsilon), & T_1 = r(x,y)+\varepsilon, & T_1 = e^{-\varepsilon} r(x,y),
\end{array}$$

where ε is a real number. Here we can find the general group of the symmetries by considering a general linear combination $c_1\mathbf{v}_1 + \cdots + c_1\mathbf{v}_4$ of the given vector fields. In particular if g is the action of the symmetry group near the identity, it can be represented in the form $g = \exp(\varepsilon_4\mathbf{v}_4) \circ \cdots \circ \exp(\varepsilon_1\mathbf{v}_1)$.

3 Symmetry reduction for steady two-dimensional boundary-layer stagnation-point flow equations

Lie-group method is applicable to both linear and non-linear partial differential equations, which leads to similarity variables that may be used to reduce the number of independent variables in partial differential equations. By determining the transformation group under which a given partial differential equation is invariant, we can obtain information about the invariants and symmetries of that equation.

The first advantage of symmetry group method is to construct new solutions from known solutions. Neither the first advantage nor the second will be investigated here, but symmetry group method will be applied to the Eq. (2) to be connected directly to some order differential equations. To do this, a particular linear combinations of infinitesimals are considered and their corresponding invariants are determined. The steady two-dimensional boundary-layer stagnation-point flow equations expressed in the coordinates (x, y), so to reduce this equation is to search for its form in specific coordinates. Those coordinates will be constructed by searching for independent invariants (r, s) corresponding to an infinitesimal generator. So using the chain rule, the expression of the equation in the new coordinate allows us to the reduced equation. Here we will obtain some invariant solutions with respect to symmetries. First we obtain the similarity variables for each term of the Lie algebra $\mathfrak{g}$, then we use this method to reduced the PDE and find the invariant solutions. Here our system has two-independent and four-dependent variables, thus, the similarity transformations and invariant functions with invariant solutions are coming in the following table.

vector field	**invariant function**	**invariant transformations**	**similarity transformations**
$\mathbf{v}_1$	$\Phi(y, U, V, P, T)$	$x = r + \varepsilon, y = s$	$x = r, y = s,$
$\mathbf{v}_2$	$\Phi(x, U, V, P, T)$	$x = r, y = s + \varepsilon$	$x = r, y = s$
$\mathbf{v}_3$	$\Phi(x, y, U, V, P)$	$T = f(r, s) + \varepsilon$	translating of all function T is the similarity transformation
$\mathbf{v}_4$	$\Phi(x, y, U, V, P)$	$T = e^{\varepsilon} f(r, s)$	scaling of all function T is the similarity transformation

In the above table, the invariant solution with respect to $\mathbf{v}_1$ and $\mathbf{v}_2$ obtained from translation on both dependent variables, and for $\mathbf{v}_3$ and $\mathbf{v}_4$ all invariant solutions are obtained by translating and scaling on function T.

4 Optimal system of steady two-dimensional boundary-layer stagnation-point flow equations

Let a system of differential equation Δ admitting the symmetry Lie group G,be given. Now G operates on the set of solutions S of Δ. Let $s \cdot G$ be the orbit of s, and H be a subgroup of G. Invariant $H-$solutions $s \in S$ are characterized by equality $s \cdot S = \{s\}$. If $h \in G$ is a transformation and $s \in S$,then $h \cdot (s \cdot H) = (h \cdot s) \cdot (hHh^{-1})$. Consequently,every invariant $H-$solution s transforms into an invariant $hHh^{-1}-$solution (Proposition 3.6 of [5]).

Therefore, different invariant solutions are found from similar subgroups of G. Thus, classification of invariant $H-$solutions is reduced to the problem of classification of subgroups of G,up to similarity. An optimal system of $s-$dimensional subgroups of G is a list of conjugacy inequivalent $s-$dimensional subgroups of G with the property

that any other subgroup is conjugate to precisely one subgroup in the list. Similarly, a list of $s-$dimensional subalgebras forms an optimal system if every $s-$dimensional subalgebra of $\mathfrak{g}$ is equivalent to a unique member of the list under some element of the adjoint representation: $\tilde{\mathfrak{h}} = \mathrm{Ad}(g) \cdot \mathfrak{h}$.

Let H and $\tilde{H}$ be connected, $s-$dimensional Lie subgroups of the Lie group G with corresponding Lie subalgebras $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$ of the Lie algebra $\mathfrak{g}$ of G. Then $\tilde{H} = gHg^{-1}$ are conjugate subgroups if and only $\tilde{\mathfrak{h}} = \mathrm{Ad}(g) \cdot \mathfrak{h}$ are conjugate subalgebras (Proposition 3.7 of [5]). Thus,the problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras, and so we concentrate on it.

4.1 One-dimensional optimal system

For one-dimensional subalgebras,the classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation, since each one-dimensional subalgebra is determined by a nonzero vector in Lie algebra symmetries of steady two-dimensional boundary-layer stagnation-point flow equations and so to "simplify" it as much as possible.

The adjoint action is given by the Lie series

$$\mathrm{Ad}(\exp(\varepsilon \mathbf{v}_i)\mathbf{v}_j) = \mathbf{v}_j - \varepsilon[\mathbf{v}_i, \mathbf{v}_j] + \frac{\varepsilon^2}{2}[\mathbf{v}_i, [\mathbf{v}_i, \mathbf{v}_j]] - \cdots, \tag{5}$$

where $[\mathbf{v}_i, \mathbf{v}_j]$ is the commutator for the Lie algebra, ε is a parameter, and $i, j = 1, \cdots, 4$. Let $F_i^\varepsilon : \mathfrak{g} \to \mathfrak{g}$ defined by $\mathbf{v} \mapsto \mathrm{Ad}(\exp(\varepsilon \mathbf{v}_i)\mathbf{v})$ is a linear map, for $i = 1, \cdots, 4$. The matrices M_i^ε of F_i^ε, $i = 1, \cdots, 4$, with respect to basis $\{\mathbf{v}_1, \cdots, \mathbf{v}_4\}$ are

$$M_1^\varepsilon = M_2^\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_3^\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\varepsilon & 1 \end{pmatrix}, \quad M_4^\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^\varepsilon & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{6}$$

by acting these matrices on a vector field $\mathbf{v}$ alternatively we can show that a one-dimensional optimal system of $\mathfrak{g}$ is given by

$$X_1 = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2, \quad X_2 = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3, \tag{7}$$

where α_i's are real constants.

4.2 Two-dimensional optimal system

Next step is to construct two-dimensional optimal system, i.e., classification of two-dimensional subalgebras of $\mathfrak{g}$. The process is by selecting one of the vectors in (7), say, any vector of (7). Let us consider X_1 (or X_2). Corresponding to it, a vector field $X = a_1\mathbf{v}_1 + \cdots + a_4\mathbf{v}_4$, where a_i's are smooth functions of (x, y, U, V, P, T) is chosen, so we must have

$$[X_1, X] = \lambda X_1 + \mu X, \tag{8}$$

the equation (8) leads us to the system

$$C_{jk}^i \alpha_j a_k = \lambda a_i + \mu \alpha_i \qquad (i = 1, 2, 3, 4). \tag{9}$$

The solutions of the system (9), give one of the two-dimensional generator and the second generator is X_1 or, X_2 if selected. After the construction of all two-dimensional subalgebras, for every vector fields of (7), they need to be simplified by the action of (6) in the manner analogous to the way of one-dimensional optimal system.

Table 2
Commutation relations of $\mathfrak{g}$

dimension	1	2	3	4
	$\langle \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2\rangle$	$\langle \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2, \beta_3\mathbf{v}_1 + \beta_4\mathbf{v}_2 + \mathbf{v}_3, \rangle$	$\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\rangle$	$\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\rangle$
subalgebras	$\langle \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3\rangle$	$\langle \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2, \mathbf{v}_4\rangle$	$\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\rangle$	
		$\langle \mathbf{v}_3, \mathbf{v}_4\rangle$	$\langle \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_3\rangle$	
			$\langle \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\rangle$	

Consequently the two-dimensional optimal system of $\mathfrak{g}$ has three classes of $\mathfrak{g}$'s members combinations such as

$$\begin{array}{lll} 1) & \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2, \beta_3\mathbf{v}_1 + \beta_4\mathbf{v}_2 + \mathbf{v}_3, & \beta_1\beta_3 \neq \beta_2\beta_4, \\ 2) & \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2, \mathbf{v}_4, & \beta_1^2 + \beta_2^2 \neq 0, \\ 3) & \mathbf{v}_3, \mathbf{v}_4. & \end{array} \tag{10}$$

4.3 Three-dimensional optimal system

This system can be developed by the method of expansion of two-dimensional optimal system. For this take any two-dimensional subalgebras of (10), let us consider the first two vector fields of (10), and rename them Y_1 and Y_2, thus, we have a subalgebra with basis $\{Y_1, Y_2\}$, find a vector field $Y = a_1\mathbf{v}_1 + \cdots + a_4\mathbf{v}_4$, where a_i's are smooth functions of (x, y, U, V, P, T), such the triple $\{Y_1, Y_2, Y\}$ generates a basis of a three-dimensional algebra. For that it is necessary an sufficient that the vector field Y satisfies the equations

$$\begin{aligned} [Y_1, Y] &= \lambda_1 Y + \mu_1 Y_1 + \nu_1 Y_2, \\ [Y_2, Y] &= \lambda_2 Y + \mu_2 Y_1 + \nu_2 Y_2, \end{aligned} \tag{11}$$

and following from (11), we obtain the system

$$\begin{aligned} C^i_{jk}\beta^j_r a_k &= \lambda_1 a_i + \mu_1\beta^i_r + \nu_1\beta^i_s, & r = 1,2,\ s = 3,4,\ \alpha = 1,2,3,4, \\ C^i_{jk}\beta^j_s a_k &= \lambda_2 a_i + \mu_2\beta^i_r + \nu_2\beta^i_s, & r = 1,2,\ s = 3,4,\ \alpha = 1,2,3,4. \end{aligned} \tag{12}$$

The solutions of system (12) is linearly independent of $\{Y_1, Y_2\}$ and give a three-dimensional subalgebra. This process is used for the second two couple vector fields of (12).

Consequently the three-dimensional optimal system of $\mathfrak{g}$ is given by

$$\begin{array}{ll} 1)\ \mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3, & 2)\ \mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_4, \\ 3)\ \mathbf{v}_1,\ \mathbf{v}_3,\ \mathbf{v}_4, & 4)\ \mathbf{v}_2,\ \mathbf{v}_3,\ \mathbf{v}_4. \end{array}$$

All previous calculations lead to the table (2) for the optimal system of $\mathfrak{g}$.

5 Lie Algebra Structure

$\mathfrak{g}$ has a no any non-trivial *Levi decomposition* in the form of $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{g}_1$, because $\mathfrak{g}$ has no any non-trivial radical, i.e., if $\mathfrak{r}$ be the radical of $\mathfrak{g}$, then $\mathfrak{g} = \mathfrak{r}$.

If we want to integration an involuting distribution, the process decomposes into two steps:

- integration of the evolutive distribution with symmetry Lie algebra $\mathfrak{g}/\mathfrak{r}$, and
- integration on integral manifolds with symmetry algebra $\mathfrak{r}$.

First, applying this procedure to the radical $\mathfrak{r}$ we decompose the integration problem into two parts: the integration of the distribution with semisimple algebra $\mathfrak{g}/\mathfrak{r}$, then the integration of the restriction of distribution to the integral manifold with the solvable symmetry algebra $\mathfrak{r}$.

The last step can be performed by quadratures. Moreover, every semisimple Lie algebra $\mathfrak{g}/\mathfrak{r}$ is a direct sum of simple ones which are ideal in $\mathfrak{g}/\mathfrak{r}$. Thus, the Lie-Bianchi theorem reduces the integration problem ti evolutive distributions equipped with simple algebras of symmetries. Thus, integrating of system (2), become so much easy.

Both $\mathfrak{g}$ is a solvable, because if $\mathfrak{g}^{(1)} = \langle \mathbf{v}_i, [\mathbf{v}_i, \mathbf{v}_j] \rangle = [\mathfrak{g}, \mathfrak{g}]$, we have $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] = \langle \mathbf{v}_1, \cdots, \mathbf{v}_4 \rangle$, and $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] = \langle \mathbf{v}_3 \rangle$, so, we have a chain of ideals $\mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset \{0\}$. According to the table of the commutators we can decompose $\mathfrak{g}$ in the direct sum of $\mathfrak{g} = \mathbb{R}^2 \oplus \mathfrak{a}(1)$, where $\mathfrak{a}(1)$ is the Lie algebra of Affine transformations group $A(1)$.

6 Conclusion

In this article group classification of steady two-dimensional boundary-layer stagnation-point flow equations and the algebraic structure of the symmetry group is considered. Classification of r-dimensional subalgebra is determined by constructing r-dimensional optimal system. Some invariant objects are fined and the Lie algebra structure of symmetries is found.

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